Random Cross Ratios

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Abstract

The main result of this paper is the derivation of the probability density function of the cross ratio

$$k(a,b,c,d) = \frac{(c-a)(d-b)}{(d-a)(c-b)}$$

when the numbers a, b, c and d are independent stochastic variables with identical rectangular distribution. The probability density function of the cross ratio is needed to study the effectiveness of invariant based recognition systems. A short background into invariant based recognition is given. The use of the result is illustrated by examples and a short comment is given on generalizing the result to other distributions on a, b, c and d. Although theoretical, this result is useful in understanding algorithms based on the cross ratio, e.g. in autonomous vehicle navigation, object recognition and reconstruction from images.

1. Introduction

The cross ratio of four real numbers is

$$k(a,b,c,d) = \frac{(c-a)(d-b)}{(d-a)(c-b)}. \tag{1}$$

This cross ratio is invariant under projective transformations. Geometric invariants like the cross ratio have become increasingly popular in Computer Vision, e.g. in reconstruction and recognition algorithms. One interesting application is to use the cross ratio in autonomous navigation of vehicles, cf. [1, 2]. In this application vehicles are navigated by means of identical reflective beacons, which are placed on the walls of the factory. Since many of the beacons are collinear, the cross ratio of a collinear group of beacons is invariant with respect to the viewpoint of the vehicle. This enables fast recognition. Another use of the cross ratio is the recognition of planar features using the cross ratios of five coplanar points. Such objects are common in man-made environments. The use of invariants in recognition is briefly explained in section 2. In this short paper we address the problem of assessing the effectiveness of this approach. The main result is deriving the analytical expression for the probability density function and the cumulative distribution function of the cross ratio of four random numbers, in this case four independent numbers of identical rectangular distributions. Much to our surprise this probability function has a simple analytical expression. Some brief comments are also given on how this result can be extended to other input distributions.

2. Invariant based recognition

Object based recognition using viewpoint invariant features have become increasingly popular in computer vision, cf. [6, 7]. The recognition problem can be stated as follows.

In an image, extract features ω and try to identify them with a number of known objects $\Omega_1, \ldots, \Omega_N$. One difficulty is that the extracted image feature ω, which is a projection of some 3D object features Ω, depends in a complicated way on viewpoint and camera calibration. It is possible to test whether a given object Ω_i is compatible with image features ω. This is called *matching*. Matching is usually quite time consuming, but all known information about camera and viewing situation can be used. Similar to matching is the use of *mutual invariants*, cf. [4]. These are functions f on a object/image pair whose values $f(\Omega_i, \omega)$ are zero or close to zero if the object Ω_i is compatible with image features ω. Both matching and mutual invariants share a common disadvantage: A linear search through the list of objects $\Omega_1, \ldots, \Omega_N$ is needed.

Pure image *invariants*, on the other hand, is a way of getting around this time consuming linear search through all objects in the object data base. An image invariant is a function f on the image features only, such that $f(\omega)$ is the same for all images of one object. Using such an invariant feature, one can directly *reduce* the search of compatible objects to those Ω_i, which have invariant value *close* to that of ω. Two practical questions arise.

1. What is meant by close? How can we ensure that we do not miss the correct object within some probability?
2. How much reduction is achieved?

After reducing the list of compatible objects these are examined more thoroughly using mutual invariants or matching.

One of the simplest situations is the case of four collinear points. In this case an invariant, the *cross ratio* Eq. (1), was known already by Pappus 300 A.D. Even this simple case has by itself interesting applications, e.g. in autonomous vehicle navigation [1, 2]. The analysis of the cross ratio is also needed to understand other invariants like the invariants of 5 coplanar points and the invariant of 6 non-coplanar points seen in two images. These are based on the cross ratio.

The question 1 above can be solved to reasonable degree of accuracy by standard techniques illustrated by the following example.

Example. Assume that four points $\omega = (10, 123, 223, 310)$ have been measured. Assuming that the errors have a small estimated standard deviation $\sigma_a = \sigma_b = \sigma_c = \sigma_d = 1.4$, then the cross ratio is $k(10, 123, 223, 310) \approx 1.33$ and the standard deviation σ_k of the cross ratio can be approximated by

$$\sigma_k^2 \approx (\frac{\partial k}{\partial a})^2 \sigma_a^2 + (\frac{\partial k}{\partial b})^2 \sigma_b^2 + (\frac{\partial k}{\partial c})^2 \sigma_c^2 + (\frac{\partial k}{\partial d})^2 \sigma_d^2$$

which gives $\sigma_k \approx 0.014$. So by reducing the matching to those objects with cross ratio within the interval $[k - 2.58\,\sigma_k, k + 2.58\,\sigma_k] \approx [1.29, 1.36]$, there is a low probability, approximately 0.01, that we miss the correct object. ■

The question 2 above is the question on how much reduction we can expect given such an interval. Are some cross ratios more common than others? A similar question is what happens if the image feature ω is not the projections of a known object but just four random image points. Are some cross ratios more probable than others? In the next section we try to answer this question by deriving the probability density function of four independent identically rectangularly distributed points. The reason for using rectangular distribution is based on the assumption that a random point in the image has uniform distribution over the whole screen.

3. Main Result

Theorem 3.1. *If A, B, C and D are independent stochastic variables with identical rectangular distributions, then the probability density function $f_X(x)$ of the cross ratio*

$$X = \frac{(C-A)\,(D-B)}{(D-A)\,(C-B)}$$

is

$$f_X(x) = \begin{cases} f_1(x) + f_3(x) & \text{if } x < 0 \\ f_3(x) + f_2(x) & \text{if } 0 < x < 1 \\ f_2(x) + f_1(x) & \text{if } 1 < x \end{cases} \tag{2}$$

where

$$\begin{aligned} f_1(x) &= \tfrac{1}{3}((2x-1)\ln(\tfrac{x}{x-1}) - 2)) \\ f_2(x) &= \tfrac{1}{3}(\tfrac{(x+1)\ln(x)+2(1-x)}{(x-1)^3}) \\ f_3(x) &= \tfrac{1}{3}(\tfrac{(x-2)\ln(1-x)-2x}{x^3}) \end{aligned}$$

The corresponding cumulative distribution function is

$$F_X(x) = P(X < x) = \begin{cases} F_1(x) + F_3(x) & \text{if } x < 0 \\ 1/3 & \text{if } x = 0 \\ 1/2 + F_2(x) + F_3(x) & \text{if } 0 < x < 1 \\ 2/3 & \text{if } x = 1 \\ 1 + F_1(x) + F_2(x) & \text{if } 1 < x \end{cases} \tag{3}$$

where

$$\begin{aligned} F_1(x) &= \tfrac{1}{3}(x(1-x)\ln(\tfrac{x-1}{x}) - x + \tfrac{1}{2}) \\ F_2(x) &= \tfrac{1}{3}(\tfrac{x - x\ln(x) - 1)}{(x-1)^2}) \\ F_3(x) &= \tfrac{1}{3}(\tfrac{(1-x)\ln(1-x)+x}{x^2}) \end{aligned}$$

The theorem is a consequence of the symmetry of the cross ratio with respect to permutations in $\{A, B, C, D\}$ and the following lemma.

Lemma 3.1. *If A, B, C and D are independent stochastic variables with identical rectangular distributions, then the conditional probability density function $f_{X,ordered}(x)$ of the cross ratio, given that $A < B < C < D$, is*

$$f_{X,ordered} = \begin{cases} 2((2x-1)\ln(\frac{x}{(x-1)}) - 2) & \text{if } x > 1 \\ 0 & \text{if } x \leq 1 \end{cases} \tag{4}$$

The corresponding cumulative distribution function is

$$F_{X,ordered}(x) = \begin{cases} 2\left(1 - x(x-1)\ln\left(\frac{x-1}{x}\right) - x\right) & \text{if } x > 1 \\ 0 & \text{if } x \leq 1 \end{cases} \tag{5}$$

Remark. The functions $f_1, f_2, f_3, F_1, F_2, F_3$ of Theorem 3.1 are obtained from $F_{X,\text{ordered}}$ of Lemma 3.1 by simple transformations as described later. ■

Proof. *(of the lemma)*

Since the cross ratio is invariant under projective transformations there is no loss in generality to assume that A, B, C and D are rectangularily distributed between 0 and 1, i.e.

$$f_A(a) = f_B(b) = f_C(c) = f_D(d) = \begin{cases} 0 & \text{if } x \leq 0 \quad \text{or} \quad x \geq 1 \\ 1 & \text{if } 0 < x < 1 \end{cases}$$

We will first calculate the probability $1 - F_{X,\text{ordered}}(x) = P(x < X | A < B < C < D)$, using

$$P(x < X | A < B < C < D) = \frac{P(x < X, A < B < C < D)}{P(A < B < C < D)}.$$

All of the 24 permutations are equally likely so $P(A < B < C < D) = 1/24$. Denote by P_x the joint probability $P(x < X, A < B < C < D)$, which can be calculated according to the definitions

$$P_x = \iiiint_{M_x} f_{(A,B,C,D)}(a,b,c,d)\, da\, db\, dc\, dd$$

where

$$M_x = \{(a,b,c,d)\,|\,k(a,b,c,d) > x, 0 < a < b < c < d < 1\}.$$

With this ordering the cross ratio is greater than one so $P_x = 1/24$ for all $x \leq 1$. In order to calculate the probability P_x for other x we have to

I. Determine the region M_x.

II. Calculate the integral.

It turns out that the integral can be solved analytically and the solution has a remarkably simple expression.

I. Determination of M_x. To obtain the boundaries of M_x we first study the boundaries for a. Assume that b, c and d are fixed and ordered. The partial derivative with respect to a is

$$\frac{\partial k}{\partial a} = -\frac{(d-b)(d-c)}{(d-a)^2 (c-b)}.$$

This is negative because of the ordering. The maximal cross ration is thus obtained for $a = 0$,

$$k_2(b,c,d) = k(0,b,c,d) = \frac{c\,(d-b)}{d\,(c-b)}$$

and the minimal cross ratio is obtained for $a = b$, which gives $k(b,b,c,d) = 1$. Now the condition $k(a,b,c,d) > x$ can only be satisfied using some $a \in (0,b)$ if $k_2(b,c,d)$ is greater than x and $a \in (0, a_{\max})$, where $a_{\max}$ solves $k(a_{\max},b,c,d) = x$. Some calculations give

$$a_{\max}(b,c,d,x) = \frac{cb - cd + xcd - xdb}{b - d + xc - xb}.$$

To ensure that k_2 is greater than x we fix c and d and study k_2 as a function of $b \in (0,c)$. The partial derivative of k_2 with respect to b is

$$\frac{\partial k_2}{\partial b} = \frac{c\,(d-c)}{d\,(c-b)^2}.$$

Since $0 \le b < c < d$ it follows that k_2 is strictly increasing. It attains the value 1 for $b = 0$ and $k_2 \to \infty$ as $b \to c$. For any $x > 1$ and all $0 < c < d < 1$ the condition $k_2(b,c,d) > x$ is verified for $b \in (b_{\min}, c)$, where

$$b_{\min}(c,d,x) = \frac{x-1}{x-c/d}\,c$$

fulfills $k_2(b_{\min}, c, d) = x$.

To recapitulate: We want to find those a, b, c and d with $0 < a < b < c < d < 1$, whose cross ratio $k(a,b,c,d)$ is greater than x. According to the above for $x > 1$ this set M_x is

$$M_x = \left\{ (a,b,c,d) \;\middle|\; \begin{array}{l} 0 < d < 1, \\ 0 < c < d, \\ b_{\min}(c,d,x) < b < c, \\ 0 < a < a_{\max}(b,c,d,x) \end{array} \right\}$$

For $x \le 1$ the region M_x is the whole region $\{\,(a,b,c,d)\,|0 < a < b < c < d < 1\}$.

II. Computation of the joint probability distribution. The probability P_x is

$$P_x = \iiiint_{M_x} f_{(A,B,C,D)}(a,b,c,d)\,da\,db\,dc\,dd.$$

Using rectangular distribution this becomes

$$P_x = \int_0^1 \int_0^d \int_{b_{\min}}^c \int_0^{a_{\max}} 1\,da\,db\,dc\,dd \tag{6}$$

Extensive calculations, by means of Maple, cf. Appendix A, give

$$P_x = \frac{1}{24}(2x(x-1)\ln(\frac{x-1}{x}) + 2x - 1).$$

Finally $F_{x,\text{ordered}} = 1 - 24\,P_x$ and $f_{x,\text{ordered}}$ is the derivative of $F_{x,\text{ordered}}$ with respect to x. This gives Eqs. (4) and (5). ■

The functions (4) and (5) are the basis on which the distribution function F_X and the probability density function f_X is built. This part only depends on the symmetry of the cross ratio.

Proof. (of the theorem)

The cross ratio for an ordered four point configuration $k(a,b,c,d)$ is related in a simple way to the cross ratio of all permutations of the four points $\{a,b,c,d\}$. The 24 permutations can be divided into 6 groups of 4 each. In each such group the cross ratio of the permuted points is obtained from the cross ratio of the ordered points by one of the following functions.

$$\begin{array}{llllll}
g_1: & (1,\infty) \to (1,\infty) & g_1(x) & = & x \\
g_2: & (1,\infty) \to (1,\infty) & g_2(x) & = & 1 + 1/(x-1) \\
g_3: & (1,\infty) \to (0,1) & g_3(x) & = & 1/x \\
g_4: & (1,\infty) \to (0,1) & g_4(x) & = & 1 - 1/x \\
g_5: & (1,\infty) \to (-\infty,0) & g_5(x) & = & 1 - x \\
g_6: & (1,\infty) \to (-\infty,0) & g_6(x) & = & -1/(x-1)
\end{array}$$

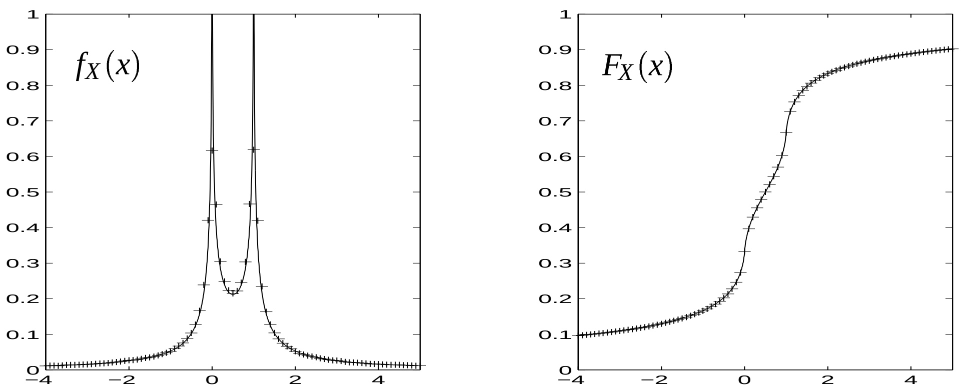

Figure 1: The probability density function and the cumulative probability distribution of the cross ratio of four points with independent identical rectangular distribution. The crosses illustrate the estimated probability function from the Monte Carlo Simulation and the graph illustrate the theoretical result.

Assume that the stochastic variable W has probability density function $f_W(w)$. A transformed stochastic variable Z, constructed as $Z = g(W)$, then has probability density function

$$f_Z(z) = f_W(g^{-1}(z)) \left| \frac{d(g^{-1})}{dz}(z) \right| \tag{7}$$

Using the transformation rule on $f_{X,\text{ordered}}$ for $g_1, \dots, g_6$ gives

$$f_X(x) = \frac{1}{6} \Sigma_{i=1}^{6} f_{X,\text{ordered}}(g_i^{-1}(x)) \left| \frac{d(g_i^{-1})}{dx}(x) \right| \tag{8}$$

which can be simplified as Eq. (2). Finally the cumulative distribution function (3) is simply is the indefinite integral of the probability density function (2). ■

Example. Continuation of the first example.

Assume as in the previous example that we have measured a cross ratio $k \approx 1.33$ and standard deviation $\sigma_k \approx 0.014$. If we have a large database of 1000 'random' objects, we can expect

$$1000\ (F_{X,\text{ordered}}(1.36) - F_{X,\text{ordered}}(1.29)) \approx 45$$

of them to have a cross ratio in the interval $[1.29, 1.36]$. Thus by using the cross ratio we directly reduce the number of objects that we have to match from 1000 to 45. These figures give an indication on the effectiveness of using the cross ratio in object recognition. ■

4. Numerical Illustration

We will now show what the probability distribution looks like and at the same time we will show results of a Monte Carlo Simulation. In the Monte Carlo simulation 300 000 samples of rectangularly distributed A, B, C and D was created using a pseudo random number generator. The cross ratios for these samples were computed. Both the distribution function and the probability function were estimated using this material. The result is shown together with a plot of the analytical expressions in Fig. 1.

5. Generalizations

In this section we will briefly discuss how the probability density function f_X of X depends on the assumed input probability density function of the four points, $f_A = f_B = f_C = f_D = f_{\mathrm{input}}$. One immediate observation is that a projective transformation of the input distribution does not change f_X since the cross ratio is invariant under projective transformations.

In this paper rectangularily distributed points have been used. This input distribution is relevant in several cases. In other cases, e.g. the cross ratio of four random directions in the plane, other distributions would be more appropriate.

The main results of this paper can easily be extended to other distributions. Typical examples are polynomial distributions with compact support. The calculations and the analytical expressions of the resulting probability function may, however, be quite complicated. Therefore, in Appendix A below, we present a Maple program that calculates the probability function of the cross ratio given a input distribution. As an example the input distribution

$$f_{\mathrm{input}}(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{if } x \le 0 \text{ or } x \ge 1 \end{cases}$$

gives

$$f_{X,\mathrm{ordered}}(x) = \big(-8(2x-1)(3x^2-3x-1)\ln(x/x-1) + 12(4x^2-4x-1)\big)/5 \qquad (9)$$

Although this is a completely different expression than Eq. (4), the general appearance of the two probability functions are quite similar.

Another input distribution, the Gaussian distribution, has been studied by Steve Maybank, cf. [5]. The appearance of the resulting probability function f_X is also quite similar to that of Eq. (4).

Thus for most of these simple input distributions the appearance of the probability density function of the cross ratio has similar appearance, i.e. similar to Fig. 1. They are different but they share some important features, having roughly the same shape with logarithmic singularities at the $0, 1$ and in some sense a logarithmic singularity also at ∞. Thus for a rough assessment of the effectiveness of using the cross ratio, any of these distributions can be used.

6. Conclusions and Acknowledgements

The main contribution of this paper is the derivation of the analytical expression for the probability function of the cross ratio of four random collinear points. It was surprising that this was possible at all and that the result was so simple. We believe that this will help understanding the effectiveness of invariant based object recognition, used for example in autonomous vehicle navigation, and surveillance systems. It seems that the general appearance of the probability function does not depend on the distribution of the four points as long as they are of independent and identical distribution. Interesting extensions of this work would be to assess the effectiveness of other invariants used in vision.

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Appendix A

This is the short maple script for calculating the distribution function $F_{x,\text{ordered}}$, for a given input distribution $f_A(x) = f_B(x) = f_C(x) = f_D(x) = f(x)$

```
# Specify probability density function on variables A,B,C,D.
f := x -> 1;  # Try for example 1 or 2*x or 3*x^2
# Specify the interval on which f is valid.
xmin := 0;
xmax := 1;

# The integration limits are xmin<d<xmax, xmin<c<d,
#                            bmin<b<c,    xmin<a<amax
amax := (c*(b-d)+kappa*d*(c-b)) / (b-d + kappa * (c-b));
bmin := c*d*(kappa-1) / (kappa*d - c);

# Integrate. Compare with Eq. (6).
slask1 := simplify(int(f(a)*f(b)*f(c)*f(d),a=xmin..amax));
slask2 := simplify(int(slask1,b=bmin..c));
slask3 := simplify(int(slask2,c=xmin..d));
F_Xordered := simplify(1-24*int(slask3,d=xmin..xmax));
f_Xordered := simplify(diff(F_Xordered,kappa));
```